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Embedding Methods for the Steady Euler Equations

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ABS: An approach to the numerical solution of the steady Euler equations is to embed the first-order Euler system in a second-order system and then to recapture the original solution by imposing additional boundary conditions. Initial development of this approach and computational experimentation with it were previously based on heuristic physical reasoning. This has led to the construction of a relaxation procedure for the solution of two-dimensional steady flow problems. The theoretical justification for the embedding approach is addressed. It is proven that, with the appropriate choice of embedding operator and additional boundary conditions, the solution to the embedded system is exactly the one to the original Euler equations. Hence, solving the embedded version of the Euler
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EMBEDDING METHODS FOR THE STEADY EULER EQUATIONS

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SUMMARY

A recent approach to the numerical solution of the steady Euler equations is to embed the first-order Euler system in a second-order system and then to recapture the original solution by imposing additional boundary conditions. Initial development of this approach and computational experimentation with it have been based on heuristic physical reasoning. This has led to the construction of a relaxation procedure for the solution of two-dimensional steady flow problems. In the present report the theoretical justification for the embedding approach is addressed. It is proven that, with the appropriate choice of embedding operator and additional boundary conditions, the solution to the embedded system is exactly the one to the original Euler equations. Hence, solving the embedded version of the Euler equations will not produce extraneous solutions.

INTRODUCTION

In the development of numerical solution procedures for the steady Euler equations, the common approach is to replace the steady equations by their unsteady counterparts and then to seek a temporally-asymptotic steady solution, either in real time ([1], [2]) or in pseudo time ([3] - [5]). Due to the difficulties associated with the numerical solution of a direct finite difference representation of the steady Euler equations, relatively few departures from this approach are to be found in the literature. Steger and Lomax [6] developed an iterative procedure for solving a nonconservation form of the steady Euler equations for subcritical flow with small shear. Desideri and Lomax [7] investigated preconditioning procedures on the matrix system arising from the finite differencing of the Euler equations. Bruneau, Chattot, Laminie and Guiu-Roux [8] have used a variational approach to transform the Euler equations into an equivalent second-order system. Preliminary numerical results have been presented for two-dimensional internal flows. Recently, Jespersen [9] has made significant progress toward adapting multigrid techniques to the solution of the Euler equations and has presented results for transonic flows over airfoils.

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Johnson [10] - [12] proposed a surrogate-equation technique, in which the first order steady Euler equations are embedded in a second order system of equations. The solution of the original Euler equations is then recaptured by imposing appropriate additional boundary conditions on the embedded system. The advantages of such an approach are that the difficulties of solving the direct difference representation of the steady Euler equations can be bypassed and the resulting second-order embedded system can be solved by a variety of well-proven numerical procedures. Initial development of this approach and computational experimentation with it have been based on heuristic physical reasoning. This has led to the construction of a relaxation procedure for the solution of two-dimensional steady flow problems. All the numerical results in [10] - [12] suggest that this is a viable and potentially more economical approach than the alternative of solving the unsteady equations of motion.

In this report the theoretical justification for the embedding approach is addressed. It is proven that, with the appropriate choice of embedding operator and additional boundary conditions, the solution to the embedded system is exactly the one to the original Euler equations. Hence, solving the embedded version of the Euler equations will not produce extraneous solutions. The following section contains the proof for the two-dimensional Euler equations and shows that for the Cauchy-Riemann equations a similar result follows immediately. Generalizations to three dimensions and to other systems of partial differential equations are mentioned subsequently.

EMBEDDING PROCEDURE

The steady Euler equations can be written in vector form as

$$f_x + g_y = 0 \tag{1}$$

where x and y are Cartesian coordinates,

$$f = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ \rho uv \\ (E + p)u \end{bmatrix} \quad \text{and} \quad g = \begin{bmatrix} \rho v \\ \rho uv \\ \rho v^2 + p \\ (E + p)v \end{bmatrix}$$

Here ρ, p, u, v, and E denote respectively the density, static pressure, velocity components in the x and y directions, and the total energy per unit volume. Furthermore

$$E = \rho\, [e + 1/2\, (u^2 + v^2)]$$

where the specific internal energy e is related to the pressure and density by the gas law

$$p = (\gamma - 1)\rho e$$

with γ denoting the ratio of specific heats.

Let

$$w = \begin{bmatrix} \rho \\ \rho u \\ \rho v \\ E \end{bmatrix}$$

By Euler's theorem on homogeneous functions, f and g can be expressed (see, for example, [13], [14]) as f = Aw and g = Bw, where A and B are the Jacobian matrices

$$A \equiv \frac{\partial f}{\partial w} \qquad \text{and} \qquad B \equiv \frac{\partial g}{\partial w}$$

We have

$$A = -\begin{bmatrix} 0 & -1 & 0 & 0 \\ \frac{3-\gamma}{2}u^2 + \frac{1-\gamma}{2}v^2 & (\gamma-3)u & (\gamma-1)v & 1-\gamma \\ uv & -v & -u & 0 \\ \frac{\gamma E u}{\rho} + (1-\gamma)u(u^2+v^2) & -\frac{\gamma E}{\rho} + \frac{\gamma-1}{2}(3u^2+v^2) & (\gamma-1)uv & -\gamma u \end{bmatrix}$$

and

$$B = -\begin{bmatrix} 0 & 0 & -1 & 0 \\ uv & -v & -u & 0 \\ \frac{3-\gamma}{2}v^2 + \frac{1-\gamma}{2}u^2 & (\gamma-1)u & (\gamma-3)v & 1-\gamma \\ \frac{\gamma v E}{\rho} + (1-\gamma)v(u^2+v^2) & (\gamma-1)uv & -\frac{\gamma E}{\rho} + \frac{\gamma-1}{2}(3v^2+u^2) & -\gamma v \end{bmatrix}$$

Now, Eq. (1) can be written as

$$\frac{\partial}{\partial x}(Aw) + \frac{\partial}{\partial y}(Bw) = 0$$

or

$$\left[\frac{\partial}{\partial x}(A\ \ ) + \frac{\partial}{\partial y}(B\ \ )\right] w = 0 \tag{2}$$

Let L denote the differential operator

$$L = \frac{\partial}{\partial x}(A\ \ ) + \frac{\partial}{\partial y}(B\ \ ) \tag{3}$$

Then the Euler equations become

$$Lw = 0 \tag{4}$$

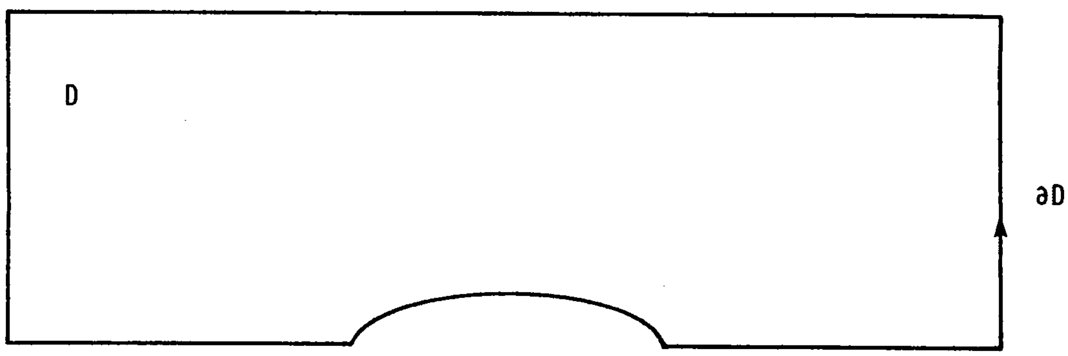
Now, let L^* be the formal adjoint operator to L defined by

$$L^* = -\left(A^T \frac{\partial}{\partial x} + B^T \frac{\partial}{\partial y}\right) \tag{5}$$

where A^T and B^T are the transposes fo A and B respectively. We may then consider the Euler equations (4) as embedded in the second-order system

$$L^* Lw = 0 \tag{6}$$

Let D be a bounded closed region with a piecewise smooth boundary. For simplicity of argument, assume that Eq. (6) is defined in a domain containing D.



We now show that with an additional condition on the boundary, ∂D, of D, solutions of Eq. (6) are also solutions of Eq. (4).

<u>Theorem</u> Let L and L^* be defined as in (3) and (5) respectively. If w is a solution of

$$L^* Lw = 0 \qquad \text{in } D$$

and satisfies the additional requirement

$$Lw = 0 \qquad \text{on } \partial D,$$

then it is also a solution of

$$Lw = 0 \qquad \text{in } D.$$

<u>Proof</u> Let $\langle \cdot , \cdot \rangle$ denote the Euclidean inner product in four-dimensional space. It can be shown that (see Appendix A for details) for any w,

$$\langle Lw, Lw \rangle - \langle w, L^*Lw \rangle = \frac{\partial}{\partial x} \langle Aw, Lw \rangle + \frac{\partial}{\partial y} \langle Bw, Lw \rangle \tag{7}$$

Integrating over D and using Green's theorem, we obtain

$$\iint_D (\langle Lw, Lw\rangle - \langle w, L^*Lw\rangle)\, dxdy$$

$$= \iint_D \left(\frac{\partial}{\partial x} \langle Aw, Lw\rangle + \frac{\partial}{\partial y} \langle Bw, Lw\rangle\right) dxdy \tag{8}$$

$$= \int_{\partial D} (\langle Aw, Lw\rangle dy - \langle Bw, Lw\rangle dx)$$

Here the line integral in the last expression of Eq. (8) is evaluated in the counterclockwise direction over the closed contour ∂D. Now, if w satisfies the hypotheses of the theorem, i.e. $L^*Lw = 0$ in D and $Lw = 0$ on ∂D, then Eq. (8) reduces to

$$\iint_D \langle Lw, Lw\rangle dxdy = 0$$

This implies that

$$\langle Lw, Lw\rangle = 0$$

in D and hence

$$Lw = 0$$

in D.

Q.E.D.

Now, consider the special case of the Cauchy-Riemann equations

$$u_x + v_y = 0 \tag{9}$$

$$v_x - u_y = 0 \tag{10}$$

Let

$$f = \begin{bmatrix} u \\ v \end{bmatrix}, \qquad g = \begin{bmatrix} v \\ -u \end{bmatrix}$$

and rewrite Eqs. (9) and (10) in vector form

$$f_x + g_y = 0 \tag{11}$$

If we choose

$$w \equiv f$$

then we have

$$A = \frac{\partial f}{\partial w} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$B = \frac{\partial g}{\partial w} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

Eq. (11) can then be expressed as

$$\frac{\partial}{\partial x}(Aw) + \frac{\partial}{\partial y}(Bw) = 0$$

or

$$\left[\frac{\partial}{\partial x}(A\quad) + \frac{\partial}{\partial y}(B\quad)\right] w = 0$$

Hence, if we again use L to denote the differential operator

$$L = \frac{\partial}{\partial x}(A\quad) + \frac{\partial}{\partial y}(B\quad) \tag{12}$$

the Cauchy-Riemann equations can also be written as

$$Lw = 0 \tag{13}$$

Let

$$L^* = -\left(A^T \frac{\partial}{\partial x} + B^T \frac{\partial}{\partial y}\right) \tag{14}$$

Then Eq. (13) can be considered as embedded in

$$L^*Lw = 0 \tag{15}$$

Note that a few simple matrix multiplications will reduce Eq. (15) to

$$\frac{\partial^2}{\partial x^2} w + \frac{\partial^2}{\partial y^2} w = 0 \tag{16}$$

which demonstrates simply the well-known fact that Eqs. (9) and (10) are embedded in the second-order system (16).

Now, let D be the same region as defined previously. Then the introduction of the differential operators L and L* for the Cauchy-Riemann equations suggests the following immediate consequence of the previous theorem.

<u>Corollary</u> If w is a solution of Eq. (16) in D and if, on the boundary of D, it satisfies Eqs. (9) and (10), then it is also a solution of Eqs. (9) and (10) in D.

Thus if one wishes to obtain the unique solution to a boundary value problem of the Cauchy-Riemann equations (9) and (10), one can also solve Eq. (16) together with the original boundary conditions and the additional requirement that Eqs. (9) and (10) be satisfied on the boundary.*

GENERALIZATIONS

Generalization of the result discussed above to the three-dimensional steady Euler equations is straightforward. Suppose the equations of motion are expressed as

$$f_x + g_y + h_z = 0$$

This can then be written as

$$\left[\frac{\partial}{\partial x}(A\quad) + \frac{\partial}{\partial y}(B\quad) + \frac{\partial}{\partial z}(C\quad)\right] q = 0$$

and hence the operator L can be introduced

$$L = \frac{\partial}{\partial x}(A\quad) + \frac{\partial}{\partial y}(B\quad) + \frac{\partial}{\partial z}(C\quad)$$

The further details are analogous to the case of two dimensions.

The embedding concept can be used on any partial differential equations expressible in the form

$$Lq = 0 \quad \text{or} \quad Lq = f$$

However, we shall not pursue this idea further here.

In Desideri and Lomax [7], preconditioning matrices are investigated. In our Eq. (6), L^* may be considered as a preconditioning operator. Hence, the embedding method is a preconditioning procedure for the continuous model, while Desideri and Lomax's approach is one for the corresponding discrete model.

Based on the mathematical formulation presented here, two-dimensional steady Euler solvers are currently being developed. A detailed description of this work, including numerical results, will be presented in a forthcoming report.

CONCLUSIONS

It has been proven that, for the numerical solution of the two-dimensional steady Euler equations, one can solve a second-order embedded system together

*The authors understand that Dr. T. N. Phillips of ICASE - NASA Langley Research Center has recently obtained a similar result for the non-homogeneous Cauchy-Riemann equations.

with appropriate additional boundary conditions. This provides a theoretical justification for the recent computational experimentation with the surrogate-equation technique.

The proof presented here is extendible to three dimensions and the embedding technique is applicable to a wider class of partial differential equations than the Euler equations of motion considered here.

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APPENDIX A

Derivation of Eq.(7) For any differentiable vector-valued functions U and V, we have

$$\left\langle \frac{\partial U}{\partial x}, V \right\rangle = - \left\langle U, \frac{\partial V}{\partial x} \right\rangle + \frac{\partial}{\partial x} \left\langle U, V \right\rangle$$

and

$$\left\langle \frac{\partial U}{\partial y}, V \right\rangle = - \left\langle U, \frac{\partial V}{\partial y} \right\rangle + \frac{\partial}{\partial y} \left\langle U, V \right\rangle$$

Hence, we have

$$\begin{aligned}
\left\langle Lw, Lw \right\rangle &= \left\langle \frac{\partial}{\partial x} (Aw) + \frac{\partial}{\partial y} (Bw), Lw \right\rangle \\
&= \left\langle \frac{\partial}{\partial x} (Aw), Lw \right\rangle + \left\langle \frac{\partial}{\partial y} (Bw), Lw \right\rangle \\
&= - \left\langle Aw, \frac{\partial}{\partial x} (Lw) \right\rangle + \frac{\partial}{\partial x} \left\langle Aw, Lw \right\rangle - \left\langle Bw, \frac{\partial}{\partial y} (Lw) \right\rangle + \frac{\partial}{\partial y} \left\langle Bw, Lw \right\rangle \\
&= - \left\langle W, A^T \frac{\partial}{\partial x} (Lw) \right\rangle + \frac{\partial}{\partial x} \left\langle Aw, Lw \right\rangle - \left\langle W, B^T \frac{\partial}{\partial y} (Lw) \right\rangle + \frac{\partial}{\partial y} \left\langle Bw, Lw \right\rangle \\
&= \left\langle W, -\left(A^T \frac{\partial}{\partial x} + B^T \frac{\partial}{\partial y}\right) LW \right\rangle + \frac{\partial}{\partial x} \left\langle Aw, Lw \right\rangle + \frac{\partial}{\partial y} \left\langle Bw, Lw \right\rangle \\
&= \left\langle W, L^*Lw \right\rangle + \frac{\partial}{\partial x} \left\langle Aw, Lw \right\rangle + \frac{\partial}{\partial y} \left\langle Bw, Lw \right\rangle
\end{aligned}$$

This is Eq. (7).

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15. Supplementary Notes

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16. Abstract

A recent approach to the numerical solution of the steady Euler equations is to embed the first-order Euler system in a second-order system and then to recapture the original solution by imposing additional boundary conditions. Initial development of this approach and computational experimentation with it have been based on heuristic physical reasoning. This has led to the construction of a relaxation procedure for the solution of two-dimensional steady flow problems. In the present report the theoretical justification for the embedding approach is addressed. It is proven that, with the appropriate choice of embedding operator and additional boundary conditions, the solution to the embedded system is exactly the one to the original Euler equations. Hence, solving the embedded version of the Euler equations will not produce extraneous solutions.

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